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The Decay of Homogeneous, Isotropic Turbulence

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ABSTRACT

Non-stationary, homogeneous, isotropic turbulence is considered from the point of view of the renormalization group. The turbulent motion is generated by a pulse of random forces. The connection between this description and that where random boundary conditions are given is made explicit. The behavior of the correlation function of the forces at small wave number is discussed. Within the linearized version of the Navier-Stokes equation, the final period of decay is governed by that correlation function. When the correlation function vanishes as k^2, the Batchelor-Proudman (1956) hypothesis of analyticity of the velocity correlation function is satisfied and $E(k,t) \sim k^4$ for small k in three space dimensions. When the correlation function is finite at $k^2 = 0$, the Saffman (1967) hypothesis of analyticity of the vorticity correlation function is satisfied and $E(k,t) \sim k^2$ for small k in three space dimensions. This is what the idea of equipartition would suggest. The renormalization group program is set up for non-stationary turbulence is set up and it is demonstrated that in both the Batchelor-Proudman and Saffman cases, the effective expansion parameter (Reynolds number) for the non-linearity of the Navier-Stokes equation vanishes as $k^2 \to 0$, so that the use of the linearized equation to study the final period of decay is justified. The renormalization group analysis necessary for the construction of the velocity correlation function $\langle v_j(\vec{x},t)v_\ell(0,\tau)\rangle$ is given in some detail.

I. INTRODUCTION

This is part of a series of papers devoted to the use of renormalization group techniques in the study of isotropic, homogeneous turbulent flow. The idea of the renormalization group is to provide a quantitative assessment of the importance of the non-linearity in the Navier-Stokes equation. This is done via an effective Reynolds number whose magnitude depends on which wave number regime one is examining.

Previously (Abarbanel, 1978 a, b) we have examined stationary, homogeneous, isotropic turbulence using renormalization group methods and found that the equations of the renormalization group allow one to reconstruct velocity correlation functions for all wave numbers using a systematic approximation scheme for the functions determining the renormalization of the viscosity and of the strength of the inertial terms in the Navier-Stokes equation. To maintain stationary homogeneous, isotropic turbulence one must drive it by random external forces (Monin and Yaglom, 1975, Section 19.6). The motion is very much determined by the properties of the correlation functions of those external forces. Several models for the small wave number variation of those correlation functions were examined both by the present author and by Forster, Nelson and Stephen (1977). We found that in each of these cases the effective Reynolds number vanishes for $k \to \infty$, the far dissipation region, and either goes to a small, finite constant as $k \to 0$ or actually goes to zero in that limit. This gave some confidence in the construction of the velocity correlation functions using the effective Reynolds number as the expansion parameter.

Here we will generate the turbulent motion by a finite length pulse of random forces and then examine the decay after the forces are turned off. The use of random forces to generate the random initial velocity distribution was discussed some time ago by Saffman (1967). He used a delta function pulse. Our use of a finite time pulse allows us to study an interesting difference in the small k behavior of the two-fold velocity correlation function between stationary and non-stationary turbulence. Furthermore, by discussing the final period of decay of turbulence we are able to relate hypotheses about the analyticity of the velocity correlation function in wave number (Batchelor and Proudman (1956) and Saffman (1967)) to the behavior of the correlation function of the random forces at small wave number. Since we will show below that in the problem of freely decaying turbulence the effective Reynolds number as $k \to 0$ goes to zero, the use of the linearized theory by these previous authors is justified and the connection is precise.

To be more explicit, we will consider random forces operating between some initial time $-T_i$ and a final time T_f. During that time these forces are δ-function correlated in time and taken to be gaussian random fields with zero mean. The correlation function is written as

$$< F_j(\vec{k}, \omega) F_\ell^*(\vec{q}, \omega') > = (2\pi)^D \delta^D(\vec{q} - \vec{k}) \frac{\gamma_0^2}{4} \Gamma_m\left(\frac{k^2}{k_0^2}\right) \int_{-T_i}^{T_f} d\tau e^{i(\omega - \omega')\tau} \Delta_{j\ell}(k) , \quad (1)$$

with $$\Delta_{j\ell}(k) = \delta_{j\ell} - k_j k_\ell / k^2 \quad , \tag{2}$$

where $F_j(\vec{k},\omega)$ is the fourier transform of the force field driving the fluid; D is the number of space dimensions; $\gamma_0^2/4$ is the strength of the correlation; and $\Gamma_m(k^2/k_0^2)$ is a dimensionless measure of the wave number variation of the correlation function on the external scale, k_0, of the mixing forces. We will show that if Saffman's hypothesis is correct, namely analytic behavior in k of the <u>vorticity</u> correlation function, then $\Gamma_m(0) \neq 0$. This is Model B of Forster, et al. and the one argued for on physical grounds by the author. If the <u>velocity</u> correlation function is analytic, then $\Gamma_m(q^2) = q^2 h_m(q^2)$ with $h_m(0) \neq 0$. This is Model A of Forster et al., and,as explained by them,in some ways is a more attractive situation physically. A discussion of the two cases will be given below.

The paper is organized as follows: In the next section I will discuss the behavior of turbulent motion generated by a random force pulse and make the connection between this description and that of giving an initial random velocity distribution. Next using the linearized Navier-Stokes equation I examine the connection of the energy spectrum function $E(k,t)$ with the correlation function of the random mixing forces generating the turbulent motion. The relation between the behavior of $\Gamma_m(k^2/k_0^2)$ near $k^2 = 0$ and the Batchelor-Proudman or Saffman hypotheses is given then. Arguments are given to support Saffman's hypothesis. In Section III perturbation theory for the construction of velocity correlation functions is given for the non-stationary situation. We will discuss the behavior of the effective expansion parameter (Reynolds number) and show that it goes to zero in the $k^2 \to 0$ limit, so that the use of the linearized Navier-Stokes equation to study the final period of decay is justified. In the 4th Section we use renormalization group methods to construct the velocity-velocity correlation function; this is an application to the present problem of the ideas in Abarbanel (1978b). Finally a summary and discussion ends the paper.

II. GENERATING THE TURBULENT MOTION

We begin by considering the Navier-Stokes equation with a given initial condition, $v_j(\vec{k}, T)$, on the velocity field at wave number $\vec{k}$ and time T. As usual the goal is to solve the equations of motion

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) v_j(\vec{k}, t) = M_{j\ell n}(\vec{k}) \int \frac{d^D p}{(2\pi)^D} v_\ell(\vec{p}, t)\, v_n(\vec{k} - \vec{p}, t) \ , \tag{3}$$

subject to the given boundary condition. In this equation ν_0 is the kinematic viscosity,

$$M_{j\ell n}(\vec{k}) = -\frac{i}{2}\left(\Delta_{j\ell}(k)\, k_n + \Delta_{jn}(k)\, k_\ell\right) , \tag{4}$$

and we are working in D space dimensions. We want to take the laplace transform now with respect to t, so let

$$w_j(\vec{p}, s) = \int_T^\infty dt\, e^{-st} v_j(\vec{p}, t) \quad , \tag{5}$$

and (3) becomes

$$(s + \nu_0 k^2) w_j(\vec{k}, s) = e^{-st} v_j(\vec{k}, T)$$

$$- M_{j\ell n}(\vec{k}) \int \frac{d^D p}{(2\pi)^D} \int_{c-i\infty}^{c+i\infty} \frac{d\sigma}{2\pi i}\, w_\ell(\vec{p}, \sigma)\, w_n(\vec{k} - \vec{p}, s - \sigma) \, , \tag{6}$$

and Re c is taken to the right of the singularities in s of $w_j(\vec{p}, s)$.

The initial velocity field has some distribution which is the embodiment of the stochastic nature of the turbulent motion. The solution of (6) will lead to expressions for the averages

$$< w_{j_1}(\vec{k}_1, s_1) \ldots w_{j_n}(\vec{k}_n, s_n) > \, , \tag{7}$$

which depend in detail on the distribution of $v_j(\vec{k}, T)$ (see Orzag (1974), Section 4.3). We wish here to replace the initial velocity distribution by a distribution of the random forces which generated the motion. This is a formal device which is useful when we want to make contact with the previous work on stationary turbulence [Abarbanel (1978 a, b), Forster, Nelson, and Stephen (1977)]. So we suppose that the random force $f_j(\vec{x}, t)$ acts over a very short time interval from, say, $t = 0$ to $t = T$. Then, following Saffman (1967) we may write

$$\frac{dv_j}{dt}(\vec{x},t) = F_j(\vec{x},t) = \left(\delta_{j\ell} - \frac{\nabla_j \nabla_\ell}{\nabla^2}\right) f_\ell \quad , \tag{8}$$

and

$$v_j(\vec{x},T) = \int_0^T dt\ F_j(\vec{x},t) . \tag{9}$$

We will take these random mixing forces to be Gaussian with zero mean and δ-function correlated in time while they act. So we choose

$$< F_j(\vec{x},t)\ F_\ell(\vec{y},\tau)> = \frac{\gamma_0^2}{4}\ \Delta_{j\ell}(\nabla)\ \tilde{\Gamma}\left(k_0^2(\vec{x}-\vec{y})^2\right)\ \delta(t-\tau) , \tag{10}$$

where

$$\Delta_{j\ell}(\nabla) = \delta_{j\ell} - \frac{1}{\nabla^2}\ \nabla_j\ \nabla_\ell \quad , \tag{11}$$

expresses the solenoidal nature of the forces which are relevant to incompressible fluids. γ_0^2 is a measure of the strength of the forces, and $\tilde{\Gamma}(k_0^2 x^2)$ tells how the forces are correlated in space. k_0^{-1} is an external length scale characteristic of the mixing forces. The correlation of the velocity field at $t = T$ is

$$< v_j(\vec{x},T)\ v_\ell(\vec{y},T)> = \frac{\gamma_0^2 T}{4}\ \Delta_{j\ell}(\nabla)\ \tilde{\Gamma}\left(k_0^2(\vec{x}-\vec{y})^2\right) \quad , \tag{12}$$

which gives the explicit relation between the distribution of initial velocities and the distribution of the mixing forces creating them. Saffman considered an infinitesimal pulse which requires us to take the limit $T \rightarrow 0$ with $\gamma_0^2 T$ held fixed.

Now that we see the equivalence of discussing turbulent motion in terms of initial velocity distributions of or random mixing forces, we will adopt the latter for the remainder of this paper. We will suppose that the force is turned on at some initial time $-T_i$ and turned off later at T_f. The stationary turbulent motion discussed previously is recovered by taking the limit $T_i, T_f \to \infty$ at the beginning. We want the forces stirring the fluid to act so they are δ-function correlated in time, and we generalize (10) to

$$< F_j(\vec{x}, t)\, F_\ell(\vec{y}, \tau)> = \frac{\gamma_0^2}{4} \Delta_{j\ell}(\nabla)\tilde{\Gamma}\left(k_0^2(\vec{x} - y)^2\right) \delta(t - \tau)\theta(t + T_i)\theta(T_f - t) . \quad (13)$$

The next question we want to address is the behavior of the mixing strength

$$\Gamma_m(k^2/k_0^2) = \int d^D x \; \tilde{\Gamma}(k_0^2 x^2)\, e^{-i\vec{k}\cdot\vec{x}} , \quad (14)$$

at small wave numbers. To look at this we assume that for long times after the mixing force has been turned off, $t >> T_f$, the turbulence has decayed sufficiently to allow us to use the linearized version of the Navier-Stokes equation. This hypothesis will be justified by the renormalization group analysis below. We want the two-fold velocity correlation function

$$\Phi_{j\ell}^{(2,0)}(t, \tau, \vec{k})$$

$$= \int d^D x \; e^{i\vec{k}\cdot\vec{x}} < v_j(\vec{x}, t)\, v_\ell(0, \tau)> \quad (15)$$

$$= \Delta_{j\ell}(k)\, \Phi^{(2,0)}(t, \tau, k^2) , \quad (16)$$

which is related to the energy spectrum function by

$$E(k,t) = \frac{(D-1)}{2} \frac{\Omega_D k^{D-1}}{(2\pi)^D} \Phi^{(2,0)}(t,t,k^2), \tag{17}$$

with $$\Omega_D = 2\pi^{D/2}/\Gamma(D/2).$$

From the linearized Navier-Stokes equation we have

$$\Phi^{(2,0)}(t,\tau,k^2) = \int \frac{d\omega\, d\omega'}{(2\pi)^2} e^{-i\omega t} e^{i\omega' t} \times$$

$$\times \frac{\gamma_0^2}{4} \Gamma_m(k^2/k_0^2) \frac{1}{(-i\omega + \nu_0 k^2)(i\omega' + \nu_0 k^2)} \int_{-T_i}^{T_f} d\rho\, e^{i(\omega - \omega')\rho}. \tag{18}$$

Let's consider two cases of this. First, the situation relevant to decay: $t > T_f$. Then from (18) we find

$$\Phi^{(2,0)}(t,t,k^2) = \frac{e^{-2\nu_0 k^2 t}}{2\nu_0 k^2} \left(e^{2\nu_0 k^2 T_f} - e^{-2\nu_0 k^2 T_i} \right) \Gamma_m\left(\frac{k^2}{k_0^2}\right) \frac{\gamma_0^2}{4} . \tag{19}$$

The second case is when $-T_i < t < T_f$, which we study to see how the linearized Navier-Stokes theory behaves for stationary turbulence. The results will be relevant only when the effective Reynolds number is small as $k^2 \to 0$. In this second case, (18) tells us

$$\Phi^{(2,0)}(t,t,k^2) = \frac{1 - e^{-2\nu_0 k^2 (t+T_i)}}{2\nu_0 k^2} \Gamma_m(k^2/k_0^2) \frac{\gamma_0^2}{4} . \tag{20}$$

In the case where $t > T_f$, the behavior of $\Phi^{(2,0)}(t,t,k^2)$ at small k^2 is determined by $\Gamma_m(k^2/k_0^2)$ alone. The factor of $1/2\nu_0 k^2$ is cancelled by the numerator. In Saffman's work where $T_i = 0$, $T_f \to 0$ with $\gamma_0^2 T_f$ fixed the same is true. If Saffman's hypothesis that the vorticity correlation function is analytic near $k = 0$ is correct, then $\Gamma_m(0)$ is finite and $E(k)$ behaves as k^{D-1}. However, if the Batchelor and Proudman hypothesis that the velocity correlation function is analytic near $k^2 = 0$ is correct, then $\Gamma_m(k^2/k_0^2)$ vanishes as k^2 near $k^2 = 0$, and we may write

$$\Gamma_m(q^2) = q^2\, h_m(q^2) \quad , \quad h_m(0) \neq 0 \, . \tag{21}$$

The case where $\Gamma_m(0) \neq 0$ has been discussed in Abarbanel (1978b) and also in Forster, et al. (1977) (their Model B). From a physical point of view there seems to be little reason to doubt the plausibility of this behavior and, indeed, the demonstration by Saffman that when $\Gamma_m(0) \neq 0$, $\gamma_0^2 \Gamma_m(0)$ is a dynamical invariant gives theoretical creedence to this idea. When we adopt the Batchelor-Proudman result, it is well known that the energy in the final period of decay behaves as $t^{-5/2}$ which has some experimental support from Batchelor and Townsend (1948). Saffman argues that those experiments on grid turbulence may well give $\Gamma_m(0) = 0$ since there is periodicity remaining in the turbulence from the periodic structure of the grid. An experiment with several parallel grids (each transverse to the flow, of course) oriented at random angles with respect to each other should eliminate this possible periodicity and allow one to decide whether the energy decays as $t^{-5/2}$ or $t^{-3/2}$ (which occurs when $\Gamma_m(0) \neq 0$).

From (20) we can now look at the situation for $k^2 \to 0$ in the stationary case by letting $T_i, T_f \to \infty$. The absence of T_f in (20) comes from causality: since $t < T_f$, the response function $\mathscr{G}^{(2,0)}$ should not know of T_f. To look at $k^2 = 0$, however, we need t large, so $T_f \to \infty$. Now

$$\Phi^{(2,0)}(t,t,k^2) = \frac{\gamma_0^2}{8\nu_0 k^2} \Gamma_m(k^2/k_0^2) \ , \tag{22}$$

and is independent of t, as it must be. The energy spectrum is

$$E(k) \sim k^{D-3} \Gamma_m(k^2/k_0^2) . \tag{23}$$

In stationary turbulent motion then the Saffman hypothesis leads to $E(k) \sim k^{D-3}$ for small k, while the Batchelor and Proudman conjecture means $E(k) \sim k^{D-1}$.

Forster, et al. (1977) point out that $E(k) \sim k^{D-1}$ is a consequence of equipartition and phase space arguments. Indeed, one expects their observation to hold when the fluid has not recently been forced and is no longer responding to the details of how it was mixed. Long into the decay period when the unforced motion has had a chance to distribute the energy among its degrees of freedom, $E(k) \sim k^{D-1}$ is natural. Under those circumstances, the arguments above lead one to prefer the Saffman hypothesis that $\Gamma_m(0) \neq 0$.

Now the energy contained in the small k region is rather small and a discussion of the $k \to 0$ behavior of $E(k)$ might appear to be much ado about very little. As noted previously, however, it has some bearing on one's ability to analyze the full k spectrum using the renormalization group and since it is experimentally accessible, merits consideration.

III. PERTURBATION THEORY AND THE RENORMALIZATION GROUP FOR NON-STATIONARY TURBULENCE

Now we would like to construct the velocity correlation functions

$$\Phi^{(n,0)}_{j_1 \cdots j_n}(\vec{x}_1,t_1,\ldots \vec{x}_n,t_n) = \langle v_{j_1}(\vec{x}_1,t_1) \ldots v_{j_n}(\vec{x}_n,t_n)\rangle \; , \tag{24}$$

in a perturbation series in the non-linear terms in the Navier-Stokes equation. Then we will use the renormalization group to assess the importance of the non-linearity in various regimes of wave number space. Some of this material is contained in Abarbanel (1978a) and the perturbation series has been given for stationary turbulence by Wyld (1961); it is repeated briefly for completeness.

To generate a perturbation series we write a generating functional for the Φ's. This involves the Lagrangian density function, $\hat{\mathscr{L}}$, for the velocity field, and the construction of $\hat{\mathscr{L}}$ requires another field $\bar{v}_j$ -- the anti-velocity. For $\hat{\mathscr{L}}$ we have

$$\begin{aligned} \hat{\mathscr{L}} = {} & \frac{1}{2}\, \bar{v}_j \frac{\overleftrightarrow{\partial}}{\partial t} v_j + \nu_0 \nabla_n \bar{v}_j \nabla_n v_j + F_j \bar{v}_j + \\ & - \frac{1}{2}\left[(\Delta_{jn}(\nabla) \;\; \nabla_\ell + \Delta_{j\ell}(\nabla)\, \nabla_n)\, \bar{v}_j \right] v_n v_\ell \;\; , \end{aligned} \tag{25}$$

with $A \overleftrightarrow{\frac{\partial}{\partial t}} B = A \frac{\partial B}{\partial t} - \frac{\partial A}{\partial t} B$ and F_j the random force. The perturbation series is derived from the generating functional for the n v_j and m $\bar{v}_j$ correlation functions. This is

$$Z[\eta_j, \bar{\eta}_j] = \int dv_j\, \delta(\nabla_n v_n)\, dv_j\, \delta(\nabla_\ell \bar{v}_\ell)\, e^{-\int(\mathscr{L} + \eta_j v_j + \bar{\eta}_j \bar{v}_j) d^D x\, dt} \times$$

$$\times\; P[F]\, dF, \tag{26}$$

where $P[F]$ is the distribution of the random forces. We will take these forces to be gaussian with zero mean and correlation function

$$< F_j(\vec{x}, t)\, F_\ell(\vec{y}, \tau) > = \frac{\gamma_0^2}{4} \delta(t - \tau) \Delta_{j\ell}(\nabla) \tilde{\Gamma}\left(k_0^2 (\vec{x} - \vec{y})^2\right) \theta(t)\, \theta(T - t) \;, \tag{27}$$

so the forces only operate over the interval $0 \le t \le T$. Since $P[F]$ is gaussian we can do the integral over F in (26) resulting in

$$Z[\eta_j, \bar{\eta}_j] = \int dv_j\, \delta(\nabla_n v_n)\, d\bar{v}_j\, \delta(\nabla_\ell \bar{v}_\ell)\, e^{-A + \int d^D x\, dt (\eta_j v_j + \bar{\eta}_j \bar{v}_j)} \;, \tag{28}$$

with

$$A = \int d^D x\, dt \left[\frac{1}{2} \bar{v}_j(\vec{x},t) \overleftrightarrow{\frac{\partial}{\partial t}} v_j(\vec{x},t) + \nu_0 \nabla_n \bar{v}_j(\vec{x},t)\, \nabla_n v_j(\vec{x},t) + \right.$$

$$+ \frac{1}{2} \bar{v}_j(\vec{x},t) \int d^D y\, \Delta_{j\ell}(\nabla) \tilde{\Gamma}\left(k_0^2 (\vec{x} - \vec{y})^2\right) \bar{v}_\ell(\vec{y},t) \theta(t)\, \theta(T - t) \frac{\gamma_0^2}{4}$$

$$\left. - \frac{1}{2} \left[(\Delta_{jn}(\nabla) \nabla_\ell + \Delta_{j\ell}(\nabla) \nabla_n) \bar{v}_j(\vec{x},t) \right] v_\ell(\vec{x},t)\, v_n(\vec{x},t) \right] . \tag{29}$$

Rescaling v_j and $\overline{v}_j$ to $\chi_j = 2\gamma_0^{-1} v_j$ and $\overline{\chi}_j = \gamma_0 \frac{1}{2} \overline{v}_j$ puts a factor γ_0 in front of the last term of (29). Now we may use $Z[\eta_j, \overline{\eta}_j]$ to give a representation for the $n\ \chi_j$ and $m\ \overline{\chi}_j$ correlation functions

$$G^{(n,m)}_{j_1 \cdots j_n, \ell_1 \cdots \ell_m}(\vec{x}_1, t_1 \cdots \vec{x}_n, t_n; \vec{y}_1, \tau_1, \cdots \vec{y}_m, \tau_m)$$

$$= \langle \chi_{j_1}(\vec{x}_1, t_1) \cdots \chi_{j_n}(\vec{x}_n, t_n) \overline{\chi}_{\ell_1}(\vec{y}_1, \tau_1) \cdots \overline{\chi}_{\ell_m}(\vec{y}_m, \tau_m) \rangle , \tag{30}$$

as a power series in γ_0.

The ingredients in this power series are the two point functions

$$G^{(1,1)}_{0\ j\ell}(\vec{k}, \omega, \omega') = \frac{1}{-i\omega' + \nu_0 k^2} \Delta_{j\ell}(k) \int_0^\infty \frac{dt}{2\pi} e^{i(\omega - \omega' + i\epsilon)t} , \tag{31}$$

and

$$G^{(2,0)}_{0\ j\ell}(\vec{k}, \omega, \omega') = \frac{\Gamma_m(k^2/k_0^2)}{(-i\omega + \nu_0 k^2)(i\omega' + \nu_0 k^2)} \Delta_{j\ell}(k) \int_0^T \frac{dt}{2\pi} e^{i(\omega - \omega')t} , \tag{32}$$

and the fusion vertex where $\chi_n(\vec{q}_1, \omega_1)$ joins with $\chi_\ell(\vec{q}_2, \omega_2)$ to form $\overline{\chi}_j(\vec{k} = \vec{q}_1 + \vec{q}_2, \omega = \omega_1 + \omega_2)$ with the factor

$$\frac{-i\gamma_0}{2(2\pi)^{\frac{D+1}{2}}} \left[\Delta_{j\alpha}(k) k_\beta + \Delta_{j\beta}(k) k_\alpha \right] \Delta_{\alpha n}(q_1) \Delta_{\beta\ell}(q_2) . \tag{33}$$

These are indicated in Fig. 1 where a solid line represents a factor of χ and a dotted line, a $\overline{\chi}$. The arrows are to keep track of the flow of

$\vec{k}$ and ω which are positive to the left. To construct $G^{(n,m)}_{j_1 \ldots \ell_m}$ draw all topologically distinct graphs made out of $G^{(1,1)}_{0\,j\ell}$, $G^{(2,0)}_{0\,j\ell}$, and the vertex (33). At each vertex conserve $\vec{q}$ and ω. Integrate $d^D q\, d\omega$ for each independent wave number and frequency. Recall that ω is not conserved across a $G^{(2,0)}$ line-this is the reflection of the non-stationarity of the problem. With each graph associate a weight unity except for closed loops containing two $G^{(2,0)}_{0\,j\ell}$ lines; these have weight 1/2.

We will require below the value of $\mathscr{G}^{(1,1)}$ defined via

$$G^{(1,1)}_{j\ell}(\vec{k},\omega,\omega') = \Delta_{j\ell}(k)\mathscr{G}^{(1,1)}(k^2,\omega,\omega') . \tag{34}$$

To lowest non-trivial order in γ_0^2 we find

$$\mathscr{G}^{(1,1)}(k^2,\omega,\omega',\gamma_0,\nu_0,k_0) = \frac{1}{-i\omega'+\nu_0 k^2}\,\frac{i}{2\pi}\,\frac{1}{\omega-\omega'+i\epsilon} +$$

$$-\frac{\gamma_0^2}{4}\,\frac{1}{-i\omega+\nu_0 k^2}\,\frac{1}{-i\omega'+\nu_0 k^2}\,\frac{e^{i(\omega-\omega')T}-1}{2\pi i(\omega-\omega')}\int\frac{d^D q}{(2\pi)^D}\,\frac{\Gamma_m(q^2/k_0^2)\,q_n\Delta_{n\ell}(k)q_\ell}{-i\omega+\nu_0\left(q^2+(\vec{q}-\vec{k})^2\right)}$$

$$\times\frac{1}{i(\omega-\omega')+2\nu_0 q^2}\left[\frac{k^2}{q^2}-\frac{2\vec{k}\cdot(\vec{k}-\vec{q})}{(D-1)(\vec{k}-\vec{q})^2}\right] . \tag{35}$$

Now the frequency dependence of the first term is singular compared to the $O(\gamma_0^2)$ or any further correction. So it is useful to smooth out the behavior by considering

$$\tilde{\mathscr{G}}^{(1,1)}(\omega, k^2, \gamma_0, \nu_0, k_0, T) = \int_{-\infty}^{+\infty} d\omega' \, \mathscr{G}^{(1,1)}(\omega, \omega', k^2, \gamma_0, \nu_0, k_0, T), \quad (36)$$

which yields from (35)

$$\tilde{\mathscr{G}}^{(1,1)}(\omega, k^2, \gamma_0, \nu_0, k_0, T)^{-1} = (-i\omega + \nu_0 k^2)\, Z_{11}\,, \quad (37)$$

where

$$Z_{11} = 1 + \frac{\gamma_0^2}{4} \, \frac{e^{i(\omega + i\nu_0 k^2)T} - 1}{i(\omega + i\nu_0 k^2)} \int \frac{d^D q}{(2\pi)^D} \, \frac{\Gamma_m(q^2/k_0^2) q_n \Delta_{n\ell}(k) q_\ell}{-i\omega + \nu_0\left(q^2 + (\vec{q} - \vec{k})^2\right)}$$

$$\times \frac{1}{-i\omega + \nu_0(2q^2 + k^2)} \left[\frac{k^2}{q^2} - \frac{2\vec{k} \cdot (\vec{k} - \vec{q})}{(D-1)(\vec{k} - \vec{q})^2} \right] . \quad (38)$$

In a similar fashion, one may construct, to the limits of patience, any $G^{(n,m)}$ to an arbitrary order in γ_0.

We turn now to the use of the renormalization group to give the $G^{(n,m)}$ as sums over pieces of every order of the expansion in γ_0. The idea is to define, at some arbitrary point in $\vec{k}, \omega$ space, new values of the viscosity, call it ν and the mixing strength, call it γ. Then we choose this point, called the renormalization point since we are renormalizing our correlation functions there, so that the effective dimensionless parameter which determines the size of the terms in the expansion in γ_0 is small. Since the point at which we renormalize the parameters of the theory is arbitrary, we must require the correlation functions to be independent of that point.

To describe the renormalization scheme we will use, let us look first at $\tilde{\mathscr{G}}^{(1,1)-1}$. When $\gamma_0 = 0$ it is just $-i\omega + \nu_0 k^2$. In general we expect the scale of the $-i\omega$ and of the $\nu_0 k^2$ to change independently. The renormalized $\tilde{\mathscr{G}}_R^{(1,1)}$ will be a function of $\omega, k^2, \gamma, \nu, k_0$, and T as well as the normalization point $\omega = i\omega_N$, $k^2 = q_N^2$. We can relate the overall magnitude of $\tilde{\mathscr{G}}_R^{(1,1)}$ and $\tilde{\mathscr{G}}^{(1,1)}$ by introducing Z_1 as

$$\tilde{\mathscr{G}}_R^{(1,1)}(\omega, k^2, \gamma, \nu, k_0, T, \omega_N, q_N^2)^{-1} = Z_1^{-1} \; \tilde{\mathscr{G}}^{(1,1)}(\omega, k^2, \gamma_0, \nu_0, k_0, T)^{-1} , \quad (39)$$

where Z_1 is independent of ω and k^2. Now require that

$$\frac{\partial}{\partial\omega} \, i \, \tilde{\mathscr{G}}_R^{(1,1)}(\omega, k^2, \dots)^{-1} \Bigg|_{\substack{\omega = i\omega_N \\ k^2 = q_N^2}} = 1 \qquad , \quad (40)$$

then Z_1 is determined by

$$Z_1 = \frac{\partial}{\partial\omega} \, i \, \tilde{\mathscr{G}}^{(1,1)}(\omega, k^2, \gamma_0, \nu_0, k_0, T)^{-1} \Bigg|_{\substack{\omega = i\omega_N \\ k^2 = q_N^2}} . \quad (41)$$

Z_1 clearly rescales the $-i\omega$ term term in $\tilde{\mathscr{G}}^{(1,1)-1}$. Rescale the $\nu_0 k^2$ term by requiring

$$\frac{\partial}{\partial k^2} \tilde{\mathscr{G}}_R^{(1,1)}(\omega, k^2, \gamma, \nu, k_0, T, \omega_N, q_N^2)^{-1}\Bigg|_{\substack{\omega = i\omega_N \\ k^2 = q_N^2}} = Z_1^{-1}\,\nu \ , \tag{42}$$

then the factor Z_ν which rescales the viscosity: $\nu = Z_\nu\,\nu_0$, is

$$Z_\nu = \frac{1}{\nu_0}\frac{\partial}{\partial k^2}\tilde{\mathscr{G}}^{(1,1)}(\omega, k^2, \gamma_0, \nu_0, k_0, T)^{-1}\Bigg|_{\substack{\omega = i\omega_N \\ k^2 = q_N^2}} \ . \tag{43}$$

To determine the rescaled mixing strength γ we turn to the fusion vertex which may be written as

$$\Delta_{j\alpha}(\vec{k})\Gamma_{\alpha,\beta\gamma}(\vec{k},\omega;\vec{q}_1,\omega_1,\vec{q}_2,\omega_2)\Delta_{\beta\eta}(\vec{q})\Delta_{\gamma\ell}(\vec{q}_2) \ , \tag{44}$$

where $\vec{k} = \vec{q}_1 + \vec{q}_2$ and $\omega = \omega_1 + \omega_2$. By looking at (33) we are led to define $\gamma = Z_\gamma\,\gamma_0$ by

$$\frac{k_\alpha \Gamma_{\alpha,\beta\gamma}\delta_{\beta\gamma}}{q_N^2}\Bigg|_{\substack{\vec{q}_1 = \vec{q}_2 = \frac{\vec{k}}{2}\,;\,k^2 = q_N^2 \\ \omega_1 = \omega_2 = \frac{\omega}{2}\,;\,\omega = i\omega_N}} = \frac{-i\gamma}{(2\pi)^{\frac{D+1}{2}}} \ . \tag{45}$$

A considerable simplification occurs if we choose $q_N = 0$. Then $Z_1 = 1$ and, to the lowest order in γ_0^2, $Z_\gamma = 1$. Only the viscosity is renormalized at this order in γ_0. The factor Z_ν is a dimensionless function of ω_N, γ, ν, k_0 and T. It can depend only on the three dimensionless variables

$$g = \frac{\gamma}{\nu^{3/2}} (k_N^2)^{D-4/4} , \tag{46}$$

$$\sigma = \frac{k_N^2}{k_0^2} , \tag{47}$$

and

$$\eta = \nu_0 k_N^2 T , \tag{48}$$

where we have introduced $k_N^2 = \omega_N/\nu$. The first of these variables is the analogue of the usual Reynolds number as discussed in some detail in Abarbanel (1978a). Carrying out the operation indicated in (33) we find to order γ_0^2

$$Z_\nu = 1 + \frac{\gamma_0^2}{\nu_0} \frac{1 - e^{-\omega_N T}}{4D(D+2)} \int \frac{d^D q}{(2\pi)^D} \Gamma_m\left(\frac{q^2}{k_0^2}\right)\left[\frac{D^2 - D - 2}{(\omega_N + 2\nu_0 q^2)^2} + \frac{4\nu_0 q^2}{(\omega_N + 2\nu_0 q^2)^3}\right], \tag{49}$$

which, by scaling q by k_N, and be shown to be a function of g, σ, and η only.

To guarantee that the physics is invariant under a change ω_N we need a condition on the way the g, σ, and η dependences of the renormalized $G_R^{(n,m)}$ are related. This is embodied in the statement

$$\omega_N \frac{\partial}{\partial\omega_N} G^{(n,m)}(\omega_i, \vec{k}_i, \gamma_0, \nu_0, k_0, T) = 0 , \tag{50}$$

which means that the full, unrenormalized $G^{(n,m)}$, which is calculated directly as a power series in γ_0 using the graphical rules given above, is independent of ω_N. With our normalization condition $Z_1 = 1$, so

$$G_R^{(n,m)}(\omega_i, \vec{k}_i, g, \nu, \sigma, \eta, \omega_N) = G^{(n,m)}(\omega_i, k_i, \gamma_0, \nu_0, k_0, T) \ , \tag{51}$$

where tensor indices are not shown. Noting that the derivative in (50) is with γ_0, ν_0, k_0, T fixed we have from the chain rule

$$\left[\omega_N \frac{\partial}{\partial\omega_N} + A\frac{\partial}{\partial g} + B\,\nu \frac{\partial}{\partial\nu} + (1 - B)\,\sigma \frac{\partial}{\partial\sigma} + (1 - B)\,\eta \frac{\partial}{\partial\eta}\right] \times$$

$$G_R^{(n,m)}(\omega_i, \vec{k}_i, g, \nu, \sigma, \eta, \omega_N) = 0, \tag{52}$$

with

$$A(g, \sigma, \eta) = \omega_N \frac{\partial}{\partial\omega_N} g \Bigg|_{\gamma_0, \nu_0, k_0, T \text{ fixed}} \tag{53}$$

$$B(g, \sigma, \eta) = \frac{\omega_N}{\nu} \frac{\partial}{\partial\omega_N} \nu \Bigg|_{\gamma_0, \nu_0, k_0, T \text{ fixed}} \tag{54}$$

$$= \omega_N \frac{\partial}{\partial\omega_N} \log Z_\nu \Bigg|_{\gamma_0, \nu_0, k_0, T \text{ fixed}} \ . \tag{55}$$

Since

$$g = \frac{\gamma}{\nu^{3/2}} (k_N^2)^{\frac{D-4}{-4}} = \frac{\gamma}{\nu^{\frac{D+2}{4}}} \omega_N^{\frac{D-4}{4}} , \tag{56}$$

$$= \frac{Z_\gamma}{Z_\nu^{\frac{D+2}{4}}} \frac{\gamma_0}{\nu_0^{\frac{D+2}{4}}} \omega_N^{\frac{D-4}{4}} , \tag{57}$$

$$A(g, \sigma, \eta) = \frac{D-4}{4} g + g\left[\omega_N \frac{\partial}{\partial\omega_N} \log Z_\gamma - \frac{D+2}{4} B(g, \sigma, \eta)\right] . \tag{58}$$

To the order we have calculated $Z_\gamma = 1$ and from (49) we learn

$$B(g, \sigma, \eta) = \frac{-g^2}{D+2} F(\sigma, \eta) , \tag{59}$$

with

$$F(\sigma, \eta) = \frac{1 - e^{-\eta}}{16D} \int \frac{d^D\ell}{(2\pi)^D} \frac{\Gamma_m(\sigma\ell^2)}{\left(\ell^2 + \frac{1}{2}\right)^4} \left[\frac{1}{2} (D^2 - D - 2) + \ell^2(D^2 - D + 1)\right]$$

$$- \frac{\eta e^{-\eta}}{16D} \int \frac{d^D\ell}{(2\pi)^D} \frac{\Gamma_m(\sigma\ell^2)}{\left(\ell^2 + \frac{1}{2}\right)^3} \left[\frac{1}{2} (D^2 - D - 2) + \ell^2 D(D - 1)\right], \tag{60}$$

$$= \frac{(1 - e^{-\eta})\sigma^{3-D/2}}{16D} \int \frac{d^D r}{(2\pi)^D} \frac{\Gamma_m(r^2)}{(r^2 + \frac{\sigma}{2})^4} \left[\frac{\sigma}{2}(D^2-D-2)+r^2(D^2-D+1)\right]$$

$$- \frac{\eta e^{-\eta}\sigma^{2-D/2}}{16D} \int \frac{d^D r}{(2\pi)^D} \frac{\Gamma_m(r^2)}{\left(r^2 + \frac{\sigma}{2}\right)^3} \left[\frac{\sigma}{2}(D^2-D-2)+r^2(D^2-D+1)\right]. \tag{61}$$

Now we want to use

$$\omega_N \frac{\partial}{\partial\omega_N} = [1 - B]\, k_N^2 \frac{\partial}{\partial k_N^2} , \tag{62}$$

and the dimensional analysis indicated in Abarbanel (1978a) to bring the derivatives from ω_N to the scale of wave numbers $\vec{k}_i$. The dimensions of $G^{(n,m)}$ are $\omega^{-1/2(m+3n)}\, k^{D/2(2-n-m)}$ leading to

$$\left[\xi \frac{\partial}{\partial\xi} - \frac{A}{1-B}\frac{\partial}{\partial g} - \frac{1}{1-B}\nu\frac{\partial}{\partial\nu} - \sigma\frac{\partial}{\partial\sigma} - \eta\frac{\partial}{\partial\eta} - \frac{D}{4}(2-n-m)\right]$$

$$G_R^{(n,m)}\left(\omega_i, \sqrt{\xi}\, k_i, g, \nu, \sigma, \eta, k_N^2\right) = 0. \tag{63}$$

This has the solution

$$G_R^{(n,m)}\left(\omega_i, \sqrt{\xi}\,\vec{k}_i, g, \nu, \sigma, \eta, k_N^2\right)$$

$$= G_R^{(n,m)}\left(\omega_i, \vec{k}_i, \tilde{g}(-\log\xi), \tilde{\nu}(-\log\xi), \tilde{\sigma}(-\log\xi), \tilde{\eta}(-\log\xi), k_N^2\right)$$

$$\times\ \xi^{(2-n-m)D/4}, \tag{64}$$

where the effective g, ν, σ, η satisfy

$$\frac{d\tilde{g}(u)}{du} = -\frac{A(\tilde{g}(u), \tilde{\sigma}(u), \tilde{\eta}(u))}{1-B(\tilde{g}(u), \tilde{\sigma}(u), \tilde{\eta}(u))}\ ;\ \tilde{g}(0) = g, \tag{65}$$

$$\frac{1}{\tilde{\nu}(u)} \frac{d\tilde{\nu}(u)}{du} = \frac{-1}{1-B(\tilde{g}(u), \tilde{\sigma}(u), \tilde{\eta}(u))} ; \; \tilde{\nu}(0) = \nu , \tag{66}$$

$$\frac{d\tilde{\sigma}(u)}{du} = - \tilde{\sigma}(u) ; \text{ so } \tilde{\sigma}(u) = \sigma e^{-u} , \tag{67}$$

and

$$\frac{d\tilde{\eta}(u)}{du} = - \tilde{\eta}(u) ; \text{ so } \tilde{\eta}(u) = \eta e^{-u} . \tag{68}$$

This is the key result of the renormalization group. It tells us how the correlation functions $G^{(n,m)}$ depend on the effective Reynolds number, $\tilde{g}$, the effective viscosity $\tilde{\nu}$, the effective external scale $\tilde{\sigma}$ and the effective pulse time $\tilde{\eta}$. The dimensionless parameter which determines the size of corrections to the unperturbed (that is, linearized) Navier-Stokes equation can be seen from (59) to be $\tilde{G}(-\log \xi) = \tilde{g}^2(-\log \xi) \; F(\xi\sigma, \xi\eta)$ at wave number ξk^2. Orzag (1974) discusses a "local" Reynolds number (see his Section 3.1) which is clearly related to our $\tilde{G}$. What is important to note is that the relative magnitude of the non-linearity in the Navier-Stokes equation is dependent on the regime of wave number space one is exploring.

The original expansion in γ_0, or more precisely in

$$g_0 = \frac{\gamma_0}{\nu_0^{3/2}} (k_N^2)^{D-4/4} , \tag{69}$$

is a very doubtful business,especially when the Reynolds number

$$R_0 = \frac{\gamma_0}{\nu_0^{3/2}} (k_0^2)^{D-4/4} , \tag{70}$$

based on the external scale, becomes large. What we have done with the machinery employed here is to provide a mapping from g_0 to another expansion parameter $\tilde{G}$ whose size will be more under control. In the next section we will discuss the construction of the important correlation function $\mathscr{G}^{(2,0)}$ and make this mapping more explicit.

Next let us turn to the behavior of $\tilde{G}(u)$. The equation for $\tilde{g}(u)$ to $o(\tilde{g}^3)$ reads

$$\frac{d\tilde{g}(u)}{du} = \frac{\epsilon}{4}\tilde{g}(u) - \frac{F(\sigma e^{-u}, \eta e^{-u})}{4}\left(1+\frac{\epsilon}{D+2}\right)\tilde{g}(u)^3 , \tag{71}$$

with $\epsilon = 4-D$. From this we find for $\tilde{G}(u) = \tilde{g}(u)^2 F(\tilde{\sigma}, \tilde{\eta})$

$$\frac{d\tilde{G}(u)}{du} = \left[\frac{\epsilon}{2} + \frac{d}{du}\log F(\tilde{\sigma},\tilde{\eta})\right]\tilde{G}(u) - \frac{1}{2}\left(1+\frac{\epsilon}{D+2}\right)\tilde{G}(u)^2 . \tag{72}$$

We can study this at large and small u by using the properties of $F(\sigma,\eta)$ exhibited in (66) and (67). The large wave number limit $\xi\to\infty$ corresponds to $u = -\log\xi \to -\infty$. In that limit $\sigma,\eta\to\infty$ and

$$F(\tilde{\sigma},\tilde{\eta}) \underset{\tilde{\sigma},\tilde{\eta}\to\infty}{\sim} \frac{\tilde{\sigma}^{-D/2}}{2D}\int \frac{d^D r}{(2\pi)^D}\Gamma_m(r^2) , \tag{73}$$

so

$$\frac{d}{du}\log F(\tilde{\sigma},\tilde{\eta}) = \frac{D}{2} , \tag{74}$$

which means

$$\tilde{G}(u) \underset{u \to -\infty}{\sim} e^{2u} \to 0 . \tag{75}$$

The effective expansion parameter $\tilde{G}(u)$ vanishes for very large wave number. That means physically that as we move into the far dissipative region the inertial term becomes totally unimportant. It provides corrections of order k^{-4} to the behavior of the linearized Navier-Stokes equation which is clearly dominated by viscous dissipation proportional to $\nu_0 k^2$. This makes excellent physical sense and is an important, albeit a bit obvious, result. As ξ is lowered (k^2 becomes smaller) we move into the inertial range and $\tilde{G}(u)$ grows. The actual magnitude of $\tilde{G}$ can be extracted from the solution to (78) which we will give shortly. We can investigate the other extreme limit $\xi \to 0$, $u = -\log \xi \to +\infty$ before that. This is the important limit for the study of the final period of decay and makes contact with the work of the previous section.

To study the k^2 small behavior of $\tilde{G}$ we need information on $\Gamma_m(k^2/k_0^2)$ near $k^2 = 0$ to determine $F(\tilde{\sigma},\tilde{\eta})$ as $u \to \infty$. We will consider the two cases discussed before:

$$\text{Case S:} \quad \Gamma_m(0) \neq 0 \tag{76}$$

$$\text{Case B - P:} \quad \Gamma_m(q^2) = q^2 h_m(q^2) \, ; \, h_m(0) \neq 0 . \tag{77}$$

In Case S

$$F(\tilde{\sigma},\tilde{\eta}) \underset{u\to+\infty}{\sim} \frac{e^{-u}\eta}{16D} \left\{ \int \frac{d^D r}{(2\pi)^D} \frac{1}{\left(r^2+\frac{1}{2}\right)^4} \left[\frac{1}{2}(D^2-D-2)+r^2(D^2-D+1)\right] \right.$$

$$\left. - \int \frac{d^D r}{(2\pi)^D} \frac{1}{\left(r^2+\frac{1}{2}\right)^3} \left[\frac{1}{2}(D^2-D-2)+r^2 D(D-1)\right] \right\}, \tag{78}$$

so

$$\frac{d\tilde{G}(u)}{du} = -\tilde{G}(u)\left(1-\frac{\epsilon}{2}\right) - \frac{1}{2}\tilde{G}(u)^2\left(1+\frac{\epsilon}{D+2}\right), \tag{79}$$

and

$$\tilde{G}(u) \underset{u\to\infty}{\sim} e^{-u\left(1-\frac{\epsilon}{2}\right)} = e^{-u(D-2)/2}. \tag{80}$$

In the limit $\xi = e^{-u} \to 0$ then, the effective expansion parameter is small for $D > 2$ and the use of the linearized Navier-Stokes equation for the final period of decay is justified. Corrections are of order $(k^2)^{(D-2)/2}$ in D space dimensions.

In Case B-P, $F(\sigma,\eta)$ behaves as

$$F(\tilde{\sigma},\tilde{\eta}) \underset{u\to\infty}{\sim} -\eta\,\sigma^{2-\frac{D}{2}} e^{-u\left(3-\frac{D}{2}\right)} \frac{D-1}{16} \int \frac{d^D r}{(2\pi)^D} \frac{h_m(r^2)}{r^2}, \tag{81}$$

so

$$\frac{d\tilde{G}(u)}{du} = -\tilde{G}(u) - \frac{\tilde{G}^2}{2}, \tag{82}$$

and

$$\tilde{G}(u) \underset{u\to\infty}{\sim} e^{-u}. \tag{83}$$

So, in this case also, the effective expansion parameter goes to zero in the low wave number limit. Again the use of the linearized Navier-Stokes theory for the study of the final period of decay is justified. In the present instance (89) tells us that corrections to the linearized results are $0(k^2)$.

Finally to trace the behavior of $\tilde{G}(u)$ between the $u \to -\infty (k^2 \to \infty)$ and $u \to +\infty (k^2 \to 0)$ limits just given, we may integrate (72) to learn

$$\tilde{G}(u) = \frac{g^2 F(\sigma e^{-u}, \eta e^{-u}) e^{\frac{\epsilon}{2} u}}{1 + \frac{g^2}{2}\left(1 + \frac{\epsilon}{D+2}\right)\int_0^u dx\, e^{\frac{\epsilon}{2} x} F\left(\sigma e^{-x}, \eta e^{-x}\right)} . \qquad (84)$$

For an appropriate choice of k_N^2, which then determines g, ν, σ, and η given the input or physical parameters γ_0, ν_0, k_0 and T, one may establish that $\tilde{G}(u)$ remains small over the whole range of u, not only at $u \to \pm\infty$ where it must vanish. Different choices for k_N^2 represent different mappings from the physical parameters to g, ν, σ and η. Since we have arranged for the physics to be invariant under changes in k_N^2 (or ω_N), we are free to choose that k_N^2 which yields the fastest convergence in a power series in $\tilde{G}(u)$.

IV. CONSTRUCTING THE VELOCITY CORRELATION FUNCTIONS

Now we wish to employ renormalization group methods to construct representations for the velocity-velocity correlation function $\mathscr{G}^{(2,0)}$ defined by

$$
\begin{aligned}
G_{j\ell}^{(2,0)}(\vec{k},\omega,\omega',\gamma_0,\nu_0,k_0,T) \\
&= \Delta_{j\ell}(k)\ \mathscr{G}^{(2,0)}(k^2,\omega,\omega',\gamma_0,\nu_0,k_0,T) \\
&= \int d^Dx\, e^{-i\vec{k}\cdot\vec{x}+i\omega t}\, e^{-i\omega'\tau} \times \\
&\quad \times \langle \chi_j(\vec{x},t)\chi_\ell(0,\tau)\rangle , \qquad (85)
\end{aligned}
$$

which is related to $\Phi^{(2,0)}$ in (16) by a factor $\gamma_0^2/4$. The general method has been outlined in Abarbanel (1978b) and Abarbanel, Bartels, Bronzan, and Sidhu (1975). The essentials will be repeated here.

We want to write $\mathscr{G}^{(2,0)}(k^2,\omega,\omega',\gamma_0,\nu_0,k_0,T)$ as

$$
\mathscr{G}^{(2,0)}(k^2,\omega,\omega',\gamma_0,\nu_0,T) = \frac{\Gamma_m(k^2/k_0^2)}{(-i\omega+\nu_0 k^2)(i\omega'+\nu_0 k^2)} \left(\int_0^T dt\, e^{i(\omega-\omega')t}\right) Z, \qquad (86)
$$

where the dimensionless factor Z is a function of the dimensionless variables:

$$
\bar{g}_0^2 = \frac{\gamma_0^2}{\nu_0^3}\,(k^2)^{\frac{D-4}{2}}, \qquad (87)
$$

$$\overline{x}_0 = \frac{\nu_0 k^2}{-i\omega} \quad , \tag{88}$$

$$\lambda = \frac{\omega'}{\omega} \quad , \tag{89}$$

$$\overline{\sigma} = \frac{k^2}{k_0^2} \quad , \tag{90}$$

and

$$\overline{\eta} = -i\omega T \ . \tag{91}$$

In perturbation theory

$$Z = 1 + \overline{g}_0^2 \, z\,(\overline{x}_0, \lambda, \overline{\sigma}, \overline{\eta}) + 0\,(\overline{g}_0^4) \ldots , \tag{92}$$

and we want to give a representation for Z which allows us to go beyond perturbation theory.

The first step is to introduce a renormalized viscosity, ν, and renormalized mixing strength, γ, with normalization conditions general enough to allow us to probe the variations in all of the parameters above. From $\mathscr{G}^{(1,1)}(k^2, \omega, \omega', \gamma_0, \nu_0, k_0, T)$ which is the scalar part of the $\chi, \overline{\chi}$ correlation function to all orders in γ_0 we make a renormalized $\mathscr{G}^{(1,1)}(k^2, \omega, \omega', g^2, \nu, \sigma, \eta)$ which is related by a scale factor Z_1 as in (39)

$$\mathscr{G}_R^{(1,1)}(k^2, \omega, \omega', g, \nu, \sigma, \eta) = Z_1 \ \mathscr{G}^{(1,1)}(k^2, \omega, \omega', \gamma_0, \nu_0, k_0, T). \tag{93}$$

Note from (35) a common factor of $(\omega-\omega')^{-1}$ in the terms of $\mathscr{G}^{(1,1)}$ and that at $\gamma_0 = 0$

$$\frac{i}{2\pi(\omega-\omega')}\;\mathscr{G}^{(1,1)\,-1} = -i\omega' + \nu_0 k^2 . \tag{94}$$

So let us require of $\mathscr{G}_R^{(1,1)}$

$$\frac{\partial}{\partial\omega'}\left[\frac{i}{2\pi(\omega-\omega')}\,\mathscr{G}_R^{(1,1)}(k^2,\omega,\omega',g,\nu,\eta,\sigma,q_N^2,\omega_N)^{-1}\right]\Bigg|_{\substack{k^2 = q_N^2 \\ \omega\ = i\omega_N \\ \omega' = \lambda i\omega_N}} , \tag{95}$$

so

$$Z_1(\lambda,g,\nu,\sigma,\eta,x) = \frac{\partial}{\partial\omega'}$$

$$\left[\frac{i}{2\pi(\omega-\omega')}\;\mathscr{G}^{(1,1)}(k^2,\omega,\omega',\gamma_0,k_0,T)^{-1}\right]\Bigg|_{\substack{k^2 = q_N^2 \\ \omega\ = i\omega_N \\ \omega' = \lambda i\omega_N}} , \tag{96}$$

where

$$g^2 = \frac{\gamma^2}{\nu^3}\,(q_N^2)^{\frac{D-4}{4}} , \tag{97}$$

$$\sigma = \frac{q_N^2}{k_0^2} , \tag{98}$$

$$\eta = \omega_N T , \tag{99}$$

and
$$x = \frac{\nu q_N^2}{\omega_N} . \tag{100}$$

The renormalized viscosity is determined by

$$\frac{\partial}{\partial k^2}\left[\frac{i}{2\pi}(\omega-\omega')^{-1}\mathscr{G}_R^{(1,1)}(k^2,\omega,\omega',g,\nu,\eta,\sigma,q_N^2,\omega_N)^{-1}\right]\Bigg|_{\substack{k^2=q_N^2\\ \omega=i\omega_N\\ \omega'=i\lambda\omega_N}} = Z_1\nu . \tag{101}$$

with $\nu = Z_\nu \nu_0$ we have

$$\frac{\partial}{\partial k^2}\left[\frac{i}{2\pi}(\omega-\omega')^{-1}\mathscr{G}^{(1,1)}(k^2,\omega,\omega',\gamma_0,\nu_0,k_0,T)^{-1}\right]\Bigg|_{\substack{k^2=q_N^2\\ \omega=i\omega_N\\ \omega'=i\lambda\omega_N}} = Z_\nu \nu_0 . \tag{102}$$

The evaluation of Z_ν requires computing $\mathscr{G}^{(1,1)}$ to some order in γ_0 and then performing the derivative indicated in (102).

Next we need a prescription for γ. We will use that given above, namely

$$\frac{k_\alpha \Gamma_{\alpha,\beta\tau}(\vec{k},\omega;\vec{q}_1,\omega_1,\vec{q}_2,\omega_2)\delta_{\beta\tau}}{k^2}\Bigg|_{\substack{k^2=0=q_1^2=q_2^2\\ \omega_1=\omega_2=\frac{i\omega_N}{2}=\frac{\omega}{2}}} = \frac{-i\gamma}{(2\pi)^{\frac{D+1}{2}}} . \tag{103}$$

As before the $0(\gamma_0^2)$ correction to γ vanishes and $\gamma = \gamma_0$ to that order with this normalization condition.

In perturbation theory Z_ν is a power series in

$$g_0^2 = \frac{\gamma_0^2}{\nu_0^3} (q_N^2)^{\frac{D-4}{2}} , \tag{104}$$

with coefficients depending on λ, σ, η, and

$$x_0 = \frac{\nu_0 q_N^2}{\omega_N} . \tag{105}$$

Z_ν, Z (evaluated at $k^2 = q_N^2$, $\omega = i\omega_N$, $\omega' = i\lambda\omega_N$) and $\mathcal{X}$, the ratio $g/g_0 = \mathcal{X} = Z_\nu^{-3/2}$, can be rewritten in terms of $g, x, \eta, \sigma, \lambda$. The connection between g and g_0 is especially important as it permits us to give for Z_ν, Z, and $\mathcal{X}$ non-perturbative expressions in g_0.

To this end we need the following renormalization group functions

$$A_\omega (g,\sigma,\eta,x,\lambda) = \omega_N \frac{\partial}{\partial \omega_N} g \Bigg|_{\gamma_0, \nu_0, k_0, T \text{ fixed}} , \tag{106}$$

$$A_q (g,\sigma,\eta,x,\lambda) = q_N^2 \frac{\partial}{\partial q_N^2} g \Bigg|_{\gamma_0, \nu_0, k_0, T \text{ fixed}} , \tag{107}$$

$$A_0 (g,\sigma,\eta,x,\lambda) = k_0^2 \frac{\partial}{\partial k_0^2} g \Bigg|_{\gamma_0, \nu_0, T \text{ fixed}} , \tag{108}$$

and $$A_T(g,\sigma,\eta,x,\lambda) = T\,\frac{\partial}{\partial T}\,g\,\bigg|_{\gamma_0,\nu_0,k_0\ \text{fixed}} . \tag{109}$$

The chain rule then yields

$$\frac{A\omega}{g} = A_\omega \frac{\partial}{\partial g}\log\mathscr{Z} + (B_\omega - 1)\,x\,\frac{\partial}{\partial x}\log\mathscr{Z} + \eta\,\frac{\partial}{\partial\eta}\log\mathscr{Z} , \tag{110}$$

$$\frac{A_q + \epsilon/4}{g} = A_q \frac{\partial}{\partial g}\log\mathscr{Z} + (1 + B_q)\,x\,\frac{\partial}{\partial x}\log\mathscr{Z} + \sigma\,\frac{\partial}{\partial\sigma}\log\mathscr{Z} , \tag{111}$$

$$\frac{A_0}{g} = A_0 \frac{\partial}{\partial g}\log\mathscr{Z} + B_0\,x\,\frac{\partial}{\partial x}\log\mathscr{Z} - \sigma\,\frac{\partial}{\partial\sigma}\log\mathscr{Z} , \tag{112}$$

and

$$\frac{A_T}{g} = A_T \frac{\partial}{\partial g}\log\mathscr{Z} + B_T\,x\,\frac{\partial}{\partial x}\log\mathscr{Z} + \eta\,\frac{\partial}{\partial\eta}\log\mathscr{Z} , \tag{113}$$

where $\epsilon = 4 - D$ and

$$(B_\omega, B_q, B_0, B_T) = \left(\omega_N \frac{\partial}{\partial\omega_N}, q_N^2\frac{\partial}{\partial q_N^2}, k_0^2\frac{\partial}{\partial k_0^2}, T\frac{\partial}{\partial T}\right)\log Z_\nu\,\bigg|_{\gamma_0,\nu_0\ \text{fixed}} . \tag{114}$$

This leads to

$$\frac{\partial}{\partial g}\log\mathscr{Z}(g,\sigma,\eta,x,\lambda) = \frac{\tilde{A} + (1 - B_\omega + B_T)\epsilon/4}{g\tilde{A}} , \tag{115}$$

with

$$\tilde{A}(g,\sigma,\eta,x,\lambda) = (A_q + A_0)(1 - B_\omega + B_T) + (A_\omega - A_T)(1 + B_0 + B_q) . \tag{116}$$

When $g = 0$, $\mathscr{Z} = 1$ so we may integrate for

$$\mathscr{X}(g,\sigma,\eta,x,\lambda) = \exp \int_0^g \frac{dg'}{g'} \left[1 + \frac{\epsilon/4\left(1-B_\omega(g',\sigma,\eta,x,\lambda)+B_T(g',\sigma,\eta,x,\lambda)\right)}{\bar{A}(g',\sigma,\eta,x,\lambda)} \right] . \quad (117)$$

In precisely the same fashion we find for Z_ν

$$Z_\nu(g,\sigma,\eta,x,\lambda) = \exp \int_0^g dg' \; \frac{\tilde{B}(g',\sigma,\eta,\lambda)}{\bar{A}(g',\sigma,\eta,\lambda)} \quad , \quad (118)$$

where

$$\tilde{B} = (B_q + B_0)(1 - B_\omega + B_T) + (B_\omega - B_T)(1 + B_0 + B_q) \quad (119)$$

$$= B_\omega + B_q + B_0 - B_T \; . \quad (120)$$

For $Z(g,\sigma,\eta,x,\lambda)$ we need the renormalization group functions

$$(C_\omega, C_q, C_0, C_T) = \left(\omega_N \frac{\partial}{\partial \omega_N}, \; q_N^2 \frac{\partial}{\partial q_N^2}, \; k_0^2 \frac{\partial}{\partial k_0^2}, \; T \frac{\partial}{\partial T} \right) \log Z \Bigg|_{\gamma_0, \nu_0 \text{ fixed}} \; . \quad (121)$$

From these we may construct

$$Z = \exp \int_0^g dg' \; \frac{\tilde{C}(g',\sigma,\eta,x,\lambda)}{\bar{A}(g',\sigma,\eta,x,\lambda)} \quad , \quad (122)$$

where

$$\tilde{C}(g,\sigma,\eta,x,\lambda) = (C_0 + C_q)(1 - B_\omega + B_T) + (C_\omega - C_T)(1 + B_0 + B_q) \; . \quad (123)$$

These equations are quite general and have as their main virtue the boundary condition that $Z, Z_\nu, \mathscr{X} \to 1$ when $g \to 0$. To proceed it is necessary to make some approximation. We will continue the spirit of our previous work and restrict ourselves to the one loop approximation for the A, B, and C functions. To that order in g^2 we have

$$\tilde{A} = -\frac{\epsilon}{4} g + a(x, \sigma, \eta, \lambda) g^3 , \tag{124}$$

$$\tilde{B} = - b(x, \sigma, \eta, \lambda) g^2 , \tag{125}$$

$$B_\omega - B_T = - \hat{b} (x, \sigma, \eta, \lambda) g^2 , \tag{126}$$

and

$$\tilde{C} = c(x, \sigma, \eta, \lambda) g^2 . \tag{127}$$

From the general expressions we then find

$$\mathscr{Z} = \left(1 - \frac{g^2}{g_1^2}\right)^{\left(1 + \frac{\hat{b}\epsilon}{4a}\right)/2} , \tag{128}$$

$$Z_\nu = \left(1 - \frac{g^2}{g_1^2}\right)^{\frac{-b}{2a}} , \tag{129}$$

and

$$Z = \left(1 - \frac{g^2}{g_1^2}\right)^{\frac{c}{2a}} , \tag{130}$$

where

$$g_1^2 (x, \sigma, \eta, \lambda) = \frac{\epsilon}{4 a(x, \sigma, \eta, \lambda)} . \tag{131}$$

Because $\mathscr{Z} = Z_\nu^{-3/2}$ at this order in g^2 with our normalization

$$a = \frac{3}{2} b - \frac{\epsilon}{4} \hat{b} , \tag{132}$$

or

$$\mathscr{X} = \left(1 - \frac{g^2}{g_1^2}\right)^{\frac{3b}{4a}} . \tag{133}$$

As shown in our previous paper the ratios a/b, $\hat{b}/a$ and c/a are independent of x, σ, η, and λ. At any given order of approximation for the renormalization group functions that will not be strictly true, however one may wish to impose it, and we will henceforth treat them as constants.

We are left with a pair of implicit equations which determine g and x as functions of γ_0 and ν_0 -- the physical parameters governing the system. Noting that g_N and ω_N are arbitrary we write

$$g = g_0 \left[1 - \frac{4 g^2 a\left(x, \frac{k^2}{k_0^2}, -i\omega T, \frac{\omega'}{\omega}\right)}{\epsilon} \right]^{3b/4a} , \tag{134}$$

and

$$x = \frac{\nu_0 k^2}{-i\omega} \left[1 - \frac{4 g^2 a\left(x, \frac{k^2}{k_0^2}, -i\omega T, \frac{\omega'}{\omega}\right)}{\epsilon} \right]^{-b/2a} , \tag{135}$$

so

$$g^2 = g_0^2 \left(\frac{\nu_0 k^2}{-i\omega x}\right)^3 , \tag{136}$$

and x is determined by

$$x = \frac{i\nu_0 k^2}{\omega}\left[1 + \frac{4i\gamma_0^2}{\epsilon\omega^3}\frac{\left(k^2\right)^{1+D/2}}{x^3} a\left(x, \frac{k^2}{k_0^2}, -i\omega T, \frac{\omega'}{\omega}\right)\right]^{-b/2a} . \tag{137}$$

Knowing x as a function of $\gamma_0, \nu_0, \overline{\sigma} = k^2/k_0^2, \overline{\eta} = -i\omega T$, and $\lambda = \omega'/\omega$ allows us to determine

$$Z = \left[\frac{i\nu_0 k^2}{\omega}\right]^{c/b} x\left(\gamma_0, \nu_0, \frac{k^2}{k_0^2}, -i\omega T, \frac{\omega'}{\omega}\right)^{-c/b} , \tag{138}$$

and thus $\mathscr{G}^{(2,0)}(k^2, \omega, \omega', \gamma_0, \nu_0, k_0, T)$ from (86). The task thus reduces to finding the function $a(x, \sigma, \eta, \lambda)$ and the ratios b/a and b/c. It is important to note that all reference to the normalization point variables q_N and ω_N and renormalized quantities has disappeared at this stage.

V. SUMMARY AND CONCLUSIONS

In this article we have treated the decay of homogeneous, isotropic turbulence using renormalization group techniques to quantitatively evaluate the importance of non-linear effects in the Navier-Stokes equations. We began by discussing the linearized Navier-Stokes equations which, following previous authors, e.g. Saffman (1967) we presumed were applicable in the final period of decay. {This procedure was a posteriori justified by the renormalization group analysis that followed.} The turbalent motion is generated by random forces, $F_j(\vec{x}, t)$, operating over a time interval $-T_i \le t \le T_f$. These we took to be gaussian with zero mean and correlation function

$$\langle F_j(\vec{x}, t)\, F_\ell(\vec{y}, \tau)\rangle = \frac{\gamma_0^2}{4}\, \Delta_{j\ell}(\nabla)\tilde{\Gamma}\left(k_0^2(\vec{x} - \vec{y})^2\right) \delta(t - \tau)\theta(T_i + t)\theta(T_f - t)\,,$$

where

$$\Delta_{j\ell}(\nabla) = \delta_{j\ell} - \nabla_j \nabla_\ell / \nabla^2 \,,$$

expresses the solenoidal aspect of the force (the fluid is incompressible, γ_0 represents the magnitude of the force, and $\tilde{\Gamma}(k_0^2 x^2)$ gives the disbribution of force-force correlations in space. The fourier transform

$$\Gamma_M(k^2/k_0^2) = \int d^D x\, e^{-i\vec{k}\cdot\vec{x}}\, \tilde{\Gamma}(k_0^2 x^2)\,,$$

plays an important role.

After making the connection between the specification of turbulent motion by giving these random forces or by giving random boundary conditions on the velocity field at $t = T_f$, we considered the behavior of the velocity correlation function

$$\Phi_{ij}^{(2,0)}(\vec{k}, t, \tau) = \int d^D x\, e^{-i\vec{k}\cdot\vec{x}} < v_i(\vec{x}, t)\, v_j(0,\tau) > ,$$

for $t, \tau > T_f$ and for $-T_i < t, \tau < T_f$. In the first case we showed, using the linearized Navier-Stokes equation, that the energy spectrum function $E(k, t)$

$$E(k,t) \propto k^{D-1} \sum_{\ell=1} \Phi_{\ell\ell}^{(2,0)}(\vec{k}, t, t) ,$$

behaves as

$$E(k, t) \underset{k^2 \to 0}{\sim} k^{D-1}\, \Gamma_M(k^2) .$$

When the velocity correlation function is analytic at $k_j = 0$, (Batchelor and Proudman (1956)), $E(k) \sim k^{D+1}$ so $\Gamma_M(k^2) \sim k^2$ for small k^2. When the vorticity correlation function is analytic, (Saffman (1967)), $E(k) \sim k^{D-1}$, so $\Gamma_M(k^2) \sim$ constant for small k^2. Since the latter is what one expects from equipartition (which should hold since we are in the final period of decay and the forcing has been off for a long time), it is physically to be preferred. Saffman also pointed out that in this case $E(k) = c\, k^{D-1} + \ldots$ with c a dynamical invariant. Now $\Gamma_M(0) \neq 0$ is just the situation expected on physical grounds in Abarbanel (1978a, b) and discussed as "Model B" by Forster, Nelson, and Stephen (1977). If it is correct, then in the final period of decay, the energy falls off as $t^{-3/2}$. An experiment was proposed to measure this which avoids the possible periodicity problems of simple grid turbulence.

After all this we turned to the renormalization group to give a quantitative estimate of the effective dimensionless expansion parameter (Reynolds number) for the series ofcorrections due to the non-linearity in the Navier-Stokes equation. We found that in each of the cases above that for $k^2 \to \infty$ the expansion parameter went rapidly to zero. This is expected and natural since in the deep dissipation region the νk^2 of dissipative effects should dominate the inertial transfer mechanism. Perhaps more surprising was that in each case the effective Reynolds number also goes rapidly to zero for $k^2 \to 0$ as long as the number of space dimensions is greater than two. With this in hand we proceeded to present the renormalization group method for constructing the full velocity correlation function. That method relies on a Reynolds number expansion of the renormalization group functions and leads to a non-perturbative determination of the correlation functions themselves. It explicitly has the correct $k^2 \to 0$ and $k^2 \to \infty$ limits and provides a systematically improvable interpolation between them.

Perhaps the main achievement here and in the previous papers, Abarbanel (1978 a, b) is the use of the renormalization group to provide a quantitative handle on the importance of the non-linearity in a stochastic system. Clearly the method is general and will find applications beyond turbulent flow. In turbulence one may in a qualitative fashion characterize the non-linearity $(\vec{v}\cdot\nabla)v_j$ as being rather "smooth" so that it becomes unimportant relative to the viscous term $\nu\nabla^2 v_j$ for large wave numbers and provides a controllable $k\to 0$ limit in physical space dimensions. It is numerically important only when the fluid is driven strongly, so v_j is large, and then, namely in the inertial range, the control one has over the $k^2\to 0$ and $k^2\to\infty$ limits allows one to use renormalization group methods as an interpolation procedure. In a more explicit fashion one may note that the actual expansion parameter is essentially $R_L^2(kL)^{-1}$, in three dimensions, with R_L the Reynolds number based on some length L. When R_L is large it is more or less equivalent to k small, which behavior is universal. A priori it is difficult to know just how general a feature this will be in non-linear stochastic problems, but one may optimistically expect it will not be uncommon.

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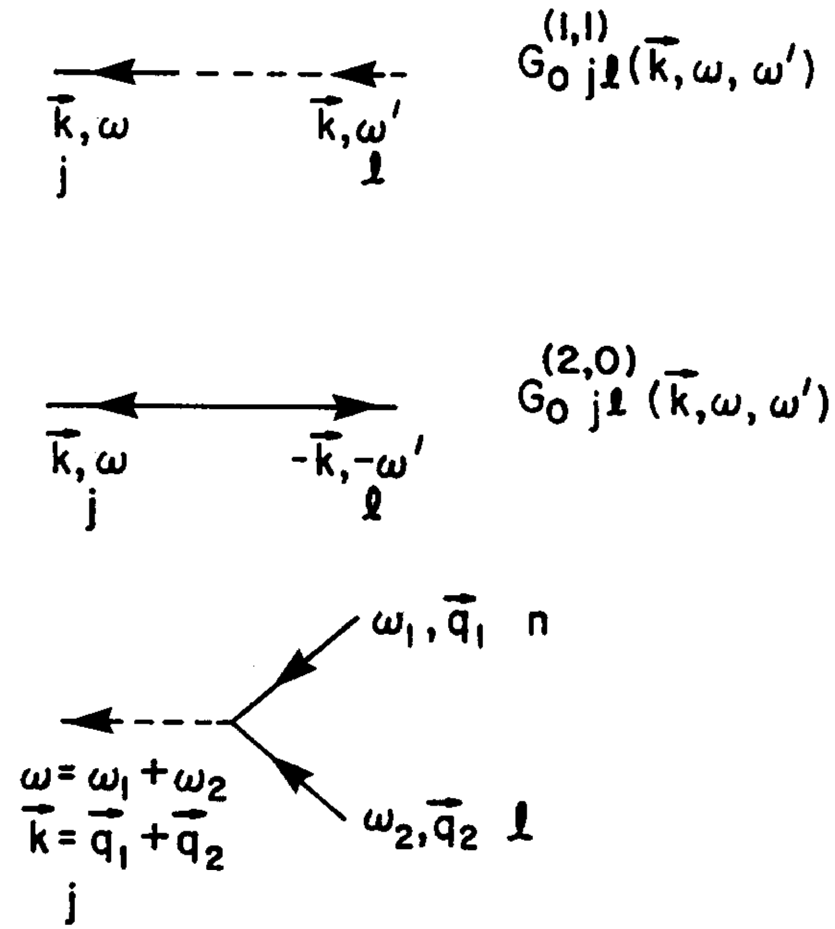
FIGURE CAPTION

Fig. 1: The elements for the graphical representation of perturbation theory in γ_0 for the velocity correlation functions. A dotted line represents an anti-velocity, $\bar{v}_j(\vec{k},\omega)$, and a solid line, a velocity $v_j(\vec{k},\omega)$. The algebraic expressions for $G_0^{(1,1)}$, $G_0^{(2,0)}$ and the fusion vertex are given in the text as equations (31), (32), and (33) respectively.



$G_{0\,j\ell}^{(1,1)}(\vec{k},\omega,\omega')$
$\vec{k},\omega$
j
$\vec{k},\omega'$
ℓ
$G_{0\,j\ell}^{(2,0)}(\vec{k},\omega,\omega')$
$\vec{k},\omega$
j
$-\vec{k},-\omega'$
ℓ
$\omega_1,\vec{q}_1$ n
$\omega_2,\vec{q}_2$ ℓ
$\omega=\omega_1+\omega_2$
$\vec{k}=\vec{q}_1+\vec{q}_2$
j

Fig. 1